A THEOREM ON ARRANGEMENTS OF LINES IN THE PLANE*

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ABSTRACT

Let $\mathscr{A}$ be an arrangement of n lines in the plane. If $R_1, \cdots, R_r$ are r distinct regions of $\mathscr{A}$, and R_i is a p_i-gon ($i = 1, \cdots, r$) then we show that

$$\sum_{i=1}^{r} p_i \leqq n + 4\binom{r}{2}.$$

Further we show that for all r this bound is the best possible if n is sufficiently large.

A collection of n distinct lines in the real projective or Euclidean plane is said to form an *arrangement* if no point of the plane belongs to all of the lines. An arrangement is said to be *simple* if each pair of lines has precisely one point in common, and no point of the plane belongs to more than two lines.

In [1; theorem 18.2.9] B. Grünbaum states and proves the following result which is attributed to N. Gunderson (see [2]).

THEOREM. *If a simple arrangement of n lines in the real projective plane contains a p-gon and a q-gon then* $p + q \leqq n + 4$.

Grünbaum also states that if a simple arrangement in the real projective plane contains a p-gon, q-gon and r-gon, then $p + q + r \leqq n + 9$. However, figure 18.2.1 of [1] shows this to be false. The following theorem corrects this statement and generalizes the result to the total number of sides contained in r distinct regions of an arrangement of n lines. The arrangements are not necessarily simple and may be in either the real projective plane, or in the Euclidean plane.

THEOREM. *Let* $\mathscr{A}$ *be an arrangement of n lines in the plane. If* $R_1, \cdots, R_r$ *are r distinct regions of* $\mathscr{A}$, *and* R_i *is a* p_i*-gon* ($i = 1, \cdots, r$) *then*

$$\sum_{i=1}^{r} p_i \leqq n + 4\binom{r}{2}. \tag{1}$$

This bound can be achieved for each r and all $n \geqq 4\binom{r}{2}$.

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Proof. It is easy to check that there are no arrangements of fine lives which contain more than one pentagon. We deduce that in any arrangement $\mathscr{A}$ there are at most four lines of $\mathscr{A}$ which contain a side of each of two distinct regions. Thus if we write $s(i,j)$ for the total number of lines of $\mathscr{A}$ which contain a side of each of the regions R_i and R_j, then

$$s(i,j) \leqq 4 \tag{2}$$

for all i, j satisfying $i \neq j$ and $1 \leqq i, j \leqq r$.

We denote by $a(i_1, \cdots, i_s)$ the number of lines of $\mathscr{A}$ which contain a side of each of the regions $R_{i_1}, \cdots, R_{i_s}$, but of none of the regions R_j, for all $j \in \{1, 2, \cdots, r\} \setminus \{i_1, \cdots, i_s\}$. Then $s(i,j)$ is the sum of all the numbers $a(i_1, \cdots, i_s)$, $(2 \leqq s \leqq r)$ which contain i, j within the parentheses, so that

$$s(i,j) = a(i,j) + \sum_{[i,j]} a(i,j,k) + \cdots + a(1,2,\cdots,r). \tag{3}$$

Here $\sum_{[i,j]} a(i_1, \cdots, i_s)$ denotes summation over all s-membered subsets $\{i_1, \cdots, i_s\}$ of $\{1, 2, \cdots, r\}$ which contain i and j.

The total number of sides of all the regions $R_1, \cdots, R_r$ is

$$\sum_{i=1}^{r} p_i . \tag{4}$$

Since a line which contains an edge of each of t of the r regions will be counted times in (4), the number of distinct lines of $\mathscr{A}$ which contain a side of at least one of the regions $R_1, \cdots, R_r$ is

$$\sum_{i=1}^{r} p_i - \sum a(i,j) - 2 \sum a(i,j,k) - \cdots - (r-1)a(1,2,\cdots,r), \tag{5}$$

where $\sum a(i_1, \cdots, i_s)$ denotes summation over all s-membered subsets $\{i_1, \cdots, i_s\}$ of $\{1, 2, \cdots, r\}$.

Since $\mathscr{A}$ is an arrangement of n lines, expression (5) has n as an upper bound, so that

$$\sum_{i=1}^{r} p_i \leqq n + \sum a(i,j) + 2 \sum a(i,j,k) + \cdots + (r-1)a(1,2,\cdots,r). \tag{6}$$

If we consider the sum $\sum s(i,j)$ over all $i \neq j$, $(1 \leqq i, j \leqq r)$, then each term, on the right-hand side of (3) yields a sum of the form

$$\sum_{[i,j]} \sum a(i_1, \cdots, i_s), \qquad (2 \leqq s \leqq r).$$

The inner summation is over all s-membered subsets $\{i_1, \cdots, i_s\}$ of $(1, 2, \cdots, r\}$ which contain i and j, and the outer sum is over all i and j satisfying $i \neq j$, $1 \leqq i, j \leqq r$. In such a sum the term $a(i_1, \cdots, i_s)$ occurs $\binom{s}{2}$ times so that

$$\sum_{[i,j]} \sum a(i_1, \cdots, i_s) = \binom{s}{2} \sum a(i_1, \cdots, i_s) \tag{7}$$

and

$$\sum s(i,j) = \sum a(i,j) + \binom{3}{2} \sum a(i,j,k) + \cdots + \binom{r}{2} a(1,2,\cdots,r). \tag{8}$$

We can solve (8) for $\sum a(i,j)$ and substituting in (6) we see that

$$\begin{aligned} \sum_{i=1}^{r} p_i \leqq n + \sum s(i,j) + \left\{2 - \binom{3}{2}\right\} \sum a(i,j,k) + \cdots \\ + \left\{(r-1) - \binom{r}{2}\right\} a(1,2,\cdots,r). \end{aligned} \tag{9}$$

The terms of the form $\left\{t - \binom{t+1}{2}\right\}$, $(2 \leqq t \leqq r-1)$, are all negative, so that

$$\sum_{i=1}^{r} p_i \leqq n + \sum s(i,j). \tag{10}$$

Using (2), we obtain

$$\sum_{i=1}^{r} p_i \leqq n + 4\binom{r}{2},$$

which is the inequality of the theorem.

To show that this is the best possible bound for each r and all $n \geqq 4\binom{r}{2}$, we construct, in the Euclidean plane, an arrangement of $4\binom{r}{2}$ lines which contains r regions, each of which is a $4(r-1)$-gon.

On a circle choose r equally spaced points $P_1, \cdots, P_r$. For each point P_i, $(1 \leqq i \leqq r)$, draw a circle, center P_i, with radius small enough to ensure that a common tangent of any pair of the small circles does not intersect or touch any of the other small circles. For each of the $\binom{r}{2}$ pairs of circles draw the four common tangents. Each small circle will be circumscribed by a $4(r-1)$-gon, there are r such regions, and we have an arrangement of $2r(r-1)$ lines such that equality in (1) is achieved. The construction in the case $r = 4$ is shown in Figure 1.

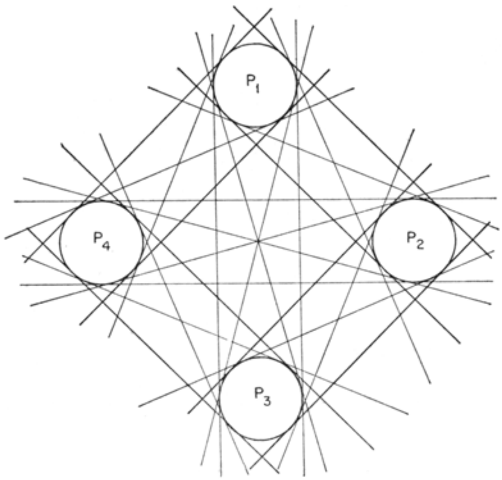

Figure 1

The Euclidean arrangement described above can be modified slightly to ensure that no two lines are parallel and no three lines are concurrent. Then we have a simple Euclidean arrangement. By embedding these arrangements in the real projective plane, we arrive at projective and simple projective arrangements. In each case we get equality in (1) so the given bound is the best possible.

It is clearly possible to introduce s further lines into the arrangements described above in such a way that each such line cuts off precisely one vertex of one of the $4(r-1)$-gons, and does not intersect any of the other $4(r-1)$-gons. The effect is to increase each side of relation (1) by s, so equality continues to hold. In this way we see that for any $r \geqq 2$ and any $n \geqq 4\binom{r}{2}$, by putting $s = n - 4\binom{r}{2}$, we can find an arrangement of n lines so that equality holds in (1).

Following Levi [3], we call a collection of simple closed curves in the projective plane, an *arrangement of pseudolines* provided

(i) Each pair of curves have precisely one point in common, and the curves cross each other at this point.

(ii) No point of the plane belongs to all the curves.

It is possible to define in the obvious way regions in arrangements of pseudolines and it is not difficult to see that there are no arrangements of 5 pseudolines which contain more than one 5-sided region. We therefore deduce that the theorem is still valid for arrangements of pseudolines.

If $n < 4\binom{r}{2}$ examples show that equality in (1) may not be possible. For example, if $r = 3$, $n = 7$, then [1; figure 18.1.1 (part 2)] shows that $p + q + r \leqq 16 = n + 9$ for all simple arrangements of 7 lines. It would be interesting to discover similar bounds for $\sum_{i=1}^{r} p_i$ when $n < 4\binom{r}{2}$.

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References

1. B. Grünbaum, *Convex Polytopes*. Wiley, London-New York-Sydney, 1967.

2. W. B. Carver, *The polygonal regions into which a plane is divided by n straight lines*, Amer. Math. Monthly **48** (1941), 667–675.

3. F. Levi, *Die Teilung der projektiven Ebene durch Gerade oder Pseudogerade*, Ber. math.-phys. Kl. sächs. Akad. Wiss. Leipzig **78** (1926), 256–267.

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